Family Name .... *Name* .... *Student ID(Matricola):* ....

Solve the problems adding to the replies short and essential explenations. *Please write the solutions in the designed areas. NO EXTRA SHEETS WILL BE ACCEPTED.* 1 Problem = 4 marks. Duration: 2 hours. No questions allowed in the first hour and in the last 20 minutes.

1	2	3a	3b	3c	3d	4	5	6	TOTAL

1. Compute the 11–adic valuation $v_{11}(100!)$.

2. Prove that if a real number has a periodic continued fraction, then it is of the form $a + b\sqrt{d}$ with $a, b \in \mathbf{Q}$, $d \in \mathbf{Z}$.

3. A positive integer d is said to be a *unitary* divisor of $n \in \mathbf{N}$ if $d|n$ and $\gcd(d, n/d) = 1$. The *number of unitary divisors* function $d^*(n) := \#\{d \in \mathbf{N} : d \text{ is a unitary divisor of n}\}$.
   a. Show that if d^* is multiplicative function.

   b. Compute a formula for $d^*(p^a)$ for all p prime and $a \in \mathbf{N}^{\geq 0}$ and deduce that, if n is square free, $d^*(n) = d(n)$. Provide an example for which $d^*(n) \neq d(n)$

   c. Consider the function $\sigma_k^*(d) = \sum_{\substack{d|n \\ d \text{ unitary divisor of } n}} d^k$ and prove that σ_k^* is multiplicative for all $k \in \mathbf{Z}$.

   d. Compute a formula for $\sigma_k^*(p^a)$ for all p prime and $a \in \mathbf{N}^{\geq 0}$ and compute $(\sigma_{-3}^* * \mu)(324)$.

4. Use the partial summation formula to produce an asymptotic formula for $\sum_{n \le T} \log^3 n$

5. State all formulas that allow to express an integer as the sum of two squares, provide some ideas of their proof and apply them to compute the number of way to write 5500 as the sum of two squares .

6. Justifying every step, prove that

$$\left(\frac{13}{p}\right)_{\mathrm{J}} = \begin{cases} 1 & \text{if } p \equiv \pm 1, \pm 3, \pm 4 \bmod 13 \\ 0 & \text{if } p = 13 \\ -1 & \text{if } p \equiv \pm 2, \pm 5, \pm 6 \bmod 13. \end{cases}$$